Hybrid integrable structure of squashed sigma models

Kentaro Yoshida (Kyoto Univ.)

Based on the collaboration with

Io Kawaguchi (Kyoto U.) and Takuya Matsumoto (U. of Sydney)

0. Introduction

- Motivation & Background -
One of the most important progress in string theory

Integrability in AdS/CFT (see Beisert et. al. 1012.3982)

Next Step: Integrable deformations of AdS/CFT

[Basso’s talk, Torrielli’s talk, Hoare’s talk, van Tongeren’s talk]

NOTE AdS/CFT belongs to the rational class (like XXX-model)

Our motive an XXZ-like deformation of AdS/CFT

As an example, we will concentrate on squashed $S^3$. 
What is squashed $S^3$?

Round $S^3$ with the radius $L$

$$ds^2 = \frac{L^2}{4} [d\theta^2 + \cos^2 \theta d\phi^2 + (d\psi + \sin \theta d\phi)^2]$$

$S^2$ \hspace{1cm} $S^1$ -fibration \hspace{1cm} 3 angles \hspace{1cm} $(\theta, \phi, \psi)$

Isometry: $SU(2)_L \times SU(2)_R$

a deformation of the round $S^3$

C=0

Squashed $S^3$

$$ds^2 = \frac{L^2}{4} [d\theta^2 + \cos^2 \theta d\phi^2 + (1 + C)(d\psi + \sin \theta d\phi)^2]$$

Isometry: $SU(2)_L \times U(1)_R$

squashing parameter
Let us introduce the $SU(2)$ group element:

$$g = e^{\phi T_1} e^{\theta T_2} e^{\psi T_3} \in SU(2)$$

Here $\theta, \phi, \psi$ are the angles of $S^3$ and $T_A$'s are the $SU(2)$ generators:

$$[T_A, T_B] = \varepsilon_{AB}^C T_C, \quad \text{Tr}(T_A T_B) = -\frac{1}{2} \delta_{AB}$$

Then the left-invariant 1-form is expanded as

$$J = g^{-1}dg = J^1 T_1 + J^2 T_2 + J^3 T_3$$

Finally the metric of squashed $S^3$ is rewritten as

$$ds^2 = \frac{L^2}{4} \left[ (J^1)^2 + (J^2)^2 + (1 + C)(J^3)^2 \right] \quad \text{(XXZ-like deformation)}$$

$$= -\frac{L^2}{2} \left[ \text{Tr}[(J)^2] - 2C(\text{Tr}(JT_3))^2 \right]$$
Global symmetry: \( SU(2)_L \times U(1)_R \)

\[
\delta^L,a g = \epsilon_L T^a g, \quad \delta^R,3 g = -\epsilon_R g T^3.
\]

Classical EOM:

\[
\partial^\mu J_\mu - 2C' \text{Tr} (T^3 J_\mu) [J^\mu, T^3] = 0
\]
We discuss the **classical integrable structure** of squashed sigma model.

For quantum integrability, see [Wiegmann, Balog-Forgacs-Palla, Basso-Rej]

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**Our claim**

There are two descriptions to describe the classical dynamics based on the global symmetry

1) **Trigonometric** description based on $U(1)_R$
2) **Rational** description based on $SU(2)_L$

Lax pair and monodromy matrix can be constructed for each of them.

1) Two kinds of Lax pairs lead to the identical EOM.
2) The monodromy matrices are gauge-equivalent.

**Hybrid integrable structure!**
Plan of the talk

1. Trigonometric description
2. Rational description
3. Equivalence of two descriptions
4. Summary & Discussions
1. Trigonometric description


The classical $r$-matrix is of trigonometric type.

\[ L_t^R(x; \lambda_R) = - \sum_{a=1}^{3} \left[ w_a(\lambda_R + \alpha) \left( J_t^a + J_x^a \right) - w_a(\lambda_R - \alpha) \left( J_t^a - J_x^a \right) \right] T^a \]

\[ L_x^R(x; \lambda_R) = - \sum_{a=1}^{3} \left[ w_a(\lambda_R + \alpha) \left( J_t^a + J_x^a \right) + w_a(\lambda_R - \alpha) \left( J_t^a - J_x^a \right) \right] T^a \]

\[ w_1(\lambda_R) = w_2(\lambda_R) = \frac{\sinh \alpha}{\sinh \lambda_R}, \quad w_3(\lambda_R) = \frac{\tanh \alpha}{\tanh \lambda_R} \quad C = -\tanh^2 \alpha \]

\[ U^R(\lambda_R) = \text{P exp} \left[ \int_{-\infty}^{\infty} dx \, L_x^R(x; \lambda_R) \right] \]

\[ \frac{d}{dt} U^R(\lambda) = 0 \]
Enhancement of $U(1)_R$

$U(1)_R$ current: $j_{\mu}^{R,3} = -2(1 + C)\text{Tr} \left( T^3 J_{\mu} \right)$

(Noether current)

$[SU(2)_R]$ is broken to $U(1)_R$ due to the squashing.

The broken components of $SU(2)_R$ are realized as non-local conserved currents.

\[ j_{\mu}^{R,\pm} = -2e^{\gamma\chi} \left( \eta_{\mu\nu} \pm i\sqrt{C}\epsilon_{\mu\nu} \right) \text{Tr} \left( T^{\pm} J^\nu \right) \]

\[ \chi(x) \equiv \frac{1}{2} \int_{-\infty}^{\infty} dy \epsilon(x - y)j_{\tau}^{R,3}(y) \]

\[ \gamma \equiv \frac{\sqrt{C}}{1 + C} \]

\[ \epsilon(x - y) \equiv \theta(x - y) - \theta(y - x) \]

Conserved charges

\[ Q^{R,3} = \int dx \ j_{t}^{R,3}(x), \quad Q^{R,\pm} = \left( \frac{\gamma}{\sinh \gamma} \right)^{1/2} \int dx \ j_{t}^{R,\pm}(x) \]
Current algebra

\[
\left\{ j_t^{R,\pm}(x), j_t^{R,\mp}(y) \right\}_P = \pm i e^{2\gamma \chi(x)} j_t^{R,3}(x) \delta(x - y)
\]

\[
= \pm \frac{i}{2\gamma} \partial_x \left[ e^{2\gamma \chi(x)} \right] \delta(x - y),
\]

\[
\left\{ j_t^{R,\pm}(x), j_t^{R,\pm}(y) \right\}_P = \pm i \gamma \epsilon(x - y) j_t^{R,\pm}(x) j_t^{R,\pm}(y),
\]

\[
\left\{ j_t^{R,3}(x), j_t^{R,\pm}(y) \right\}_P = \pm i j_t^{R,\pm}(x) \delta(x - y).
\]

\[\text{q-deformed SU}(2)_R \text{ algebra}\]

\[
\left\{ Q^{R,+}, Q^{R,-} \right\}_P = i Q^{R,3} - q^{-1} Q^{R,3}, \quad \left\{ Q^{R,3}, Q^{R,\pm} \right\}_P = \pm i Q^{R,\pm}
\]

A deformation parameter: \[q = e^\gamma = \exp\left(\frac{\sqrt{C}}{1 + C}\right)\]
There are the other non-local currents

\[ \tilde{j}^{R,\pm}_\mu \equiv -2e^{-\gamma \chi} \left( \eta_{\mu\nu} \pm i\sqrt{C} \epsilon_{\mu\nu} \right) \text{Tr} \left( T^\pm J^\nu \right) \]

\[ \tilde{Q}^{R,3} \equiv -Q^{R,3}, \quad \tilde{Q}^{R,\pm} \equiv \left( \frac{\gamma}{\sinh \gamma} \right)^{1/2} \int dx \tilde{j}^{R,\pm}_t(x) \]

c.f., the previous ones slightly different!

\[ j^{R,\pm}_\mu = -2e^{\gamma \chi} \left( \eta_{\mu\nu} \pm i\sqrt{C} \epsilon_{\mu\nu} \right) \text{Tr} \left( T^\pm J^\nu \right) \]

\[ Q^{R,3} \equiv \int dx j^{R,3}_t(x), \quad Q^{R,\pm} \equiv \left( \frac{\gamma}{\sinh \gamma} \right)^{1/2} \int dx j^{R,\pm}_t(x) \]
The defining relations of quantum affine algebra:

\[
\{ Q^{R,\pm}, Q^{R,\mp} \}_p = \pm i \frac{q Q^{R,3} - q^{-1} Q^{R,3}}{q - q^{-1}},
\]

\[
\{ \widetilde{Q}^{R,\pm}, \widetilde{Q}^{R,\mp} \}_p = \pm i \frac{q \widetilde{Q}^{R,3} - q^{-1} \widetilde{Q}^{R,3}}{q - q^{-1}},
\]

\[
\{ Q^{R,\pm}, Q^{R,3} \}_p = \mp i Q^{R,\pm}, \quad \{ Q^{R,\pm}, \widetilde{Q}^{R,3} \}_p = \pm i Q^{R,\pm},
\]

\[
\{ \widetilde{Q}^{R,\pm}, \widetilde{Q}^{R,3} \}_p = \mp i \widetilde{Q}^{R,\pm}, \quad \{ \widetilde{Q}^{R,\pm}, Q^{R,3} \}_p = \pm i \widetilde{Q}^{R,\pm},
\]

The classical analog of Serre relations:

\[
\{ Q^{R,\pm}, \{ Q^{R,\pm}, Q^{R,\pm}, \widetilde{Q}^{R,\pm} \}_p \}_p = -\gamma^2 \{ Q^{R,\pm}, \widetilde{Q}^{R,\pm} \}_p (Q^{R,\pm})^2,
\]

\[
\{ \widetilde{Q}^{R,\pm}, \{ \widetilde{Q}^{R,\pm}, \widetilde{Q}^{R,\pm}, Q^{R,\pm} \}_p \}_p = -\gamma^2 \{ \widetilde{Q}^{R,\pm}, Q^{R,\pm} \}_p (\widetilde{Q}^{R,\pm})^2.
\]
Infinite tower of conserved charges

Rule:

\[ \{ Q_{R,0}^+, \} \]
\[ \{ Q_{R,-}, \} \]

where

\[ Q_{(0)}^{R,3} = Q_{(0)}^{R,3}, \quad Q_{(1)}^{R,\pm} = Q_{(1)}^{R,\pm}, \quad Q_{(1)}^{R,\pm} = Q_{(1)}^{R,\pm} \]
Concrete expressions of conserved charges

\[ Q_{(0)}^{R,3} = \int_{-\infty}^{\infty} dx \ j_t^{R,3}(x) = -\bar{Q}_{(0)}^{R,3}, \]

\[ Q_{(1)}^{R,+} = \int_{-\infty}^{\infty} dx \ j_t^{R,+}(x), \quad \bar{Q}_{(1)}^{R,-} = \int_{-\infty}^{\infty} dx \ j_t^{R,-}(x), \]

\[ Q_{(2)}^{R,3} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \epsilon(x-y) j_t^{R,+}(x) \bar{j}_t^{R,-}(y) - 2i \int_{-\infty}^{\infty} dx j_x^{R,3}(x) - \frac{1 - C}{\sqrt{C}} Q_{(0)}^{R,3}, \]

\[ \vdots \]

\[ Q_{(1)}^{R,-} = \int_{-\infty}^{\infty} dx \ j_t^{R,-}(x), \quad \bar{Q}_{(1)}^{R,+} = \int_{-\infty}^{\infty} dx \ j_t^{R,+}(x), \]

\[ \bar{Q}_{(2)}^{R,3} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \epsilon(x-y) j_t^{R,-}(x) \bar{j}_t^{R,+}(y) + 2i \int_{-\infty}^{\infty} dx j_x^{R,3}(x) + \frac{1 - C}{\sqrt{C}} \bar{Q}_{(0)}^{R,3}, \]

\[ \vdots \]
Another derivation: expanding the monodromy matrix

Expansion in $|z_R| > 1$ → $Q^{R,3}_{(0)}, Q^{R,+}_{(1)}, \tilde{Q}^{R,-}_{(1)}, Q^{R,3}_{(2)}, \ldots$

$z_R \equiv e^{-\lambda_R}$

Expansion in $|z_R| < 1$ → $\tilde{Q}^{R,3}_{(0)}, Q^{R,-}_{(1)}, \tilde{Q}^{R,+}_{(1)}, \tilde{Q}^{R,3}_{(2)}, \ldots$

(Yangian limit)

$$\lim_{C \to 0} \frac{1}{2i \sqrt{C}} \left( Q^{R,+}_{(1)} - \tilde{Q}^{R,+}_{(1)} \right) = \int dx \ J^+_x(x) - \frac{i}{2} \iint dx dy \ \epsilon(x-y) \ J^+_t(x) J^3_t(y)$$

$$\lim_{C \to 0} \frac{1}{2i \sqrt{C}} \left( \tilde{Q}^{R,-}_{(1)} - Q^{R,-}_{(1)} \right) = \int dx \ J^-_x(x) + \frac{i}{2} \iint dx dy \ \epsilon(x-y) \ J^-_t(x) J^3_t(y)$$

$$\lim_{C \to 0} \frac{i}{4} \left( Q^{R,3}_{(2)} - \tilde{Q}^{R,3}_{(2)} \right) = \int dx \ J^3_x(x) + \frac{i}{2} \iint dx dy \ \epsilon(x-y) \ J^+_t(x) J^-_t(y)$$

$SU(2)_R$ Yangian generators are reproduced.
2. Rational description


If the conserved current \( j_{\mu} \) satisfies the flatness condition, 
\[
\epsilon^{\mu\nu}(\partial_\mu j_\nu - j_\mu j_\nu) = 0
\]
then an infinite number of conserved non-local charges can be constructed.

The flatness condition for the \( SU(2)_L \) current:
\[
\epsilon^{\mu\nu}(\partial_\mu j^L_\nu - j^L_\mu j^L_\nu) = -C \epsilon^{\mu\nu} \partial_\mu (gT_3 g^{-1}) \partial_\nu (gT_3 g^{-1})
\]
Non-vanishing, because \( C \neq 0 \). But total derivative!
Improved currents

\[ j_{\mu}^{L \pm} \equiv j_{\mu}^{L} \mp \sqrt{C} \, \epsilon_{\mu \nu} \partial^\nu (g T_3 g^{-1}) \]

Ambiguity to add a topological term

These currents satisfy the flatness condition

\[ \epsilon^{\mu \nu} \left( \partial_{\mu} j_{\nu}^{L \pm} - j_{\mu}^{L \pm} j_{\nu}^{L \pm} \right) = 0 \]

An infinite number of non-local charges (straightforward)
An infinite number of conserved non-local charges:

0-th \[ Q^{A}_{(0)} = \int dx \, j_{t}^{L_0, A}(x) \] (\(SU(2)\) Noether charge)

1-st \[ Q^{A}_{(1)} = \int dx \, j_{x}^{L_0, A}(x) + \frac{1}{4} \int \int dx dy \, \epsilon(x - y) \epsilon_{BC}^{A} \, j_{t}^{L_0, B}(x) j_{t}^{L_0, C}(y) \]

\[ \epsilon(x - y) \equiv \theta(x - y) - \theta(y - x) \]

NOTE the sign of improvement is irrelevant at the charge level

Two copies of infinite sets of conserved charges

What is the charge algebra?
Current algebra

\[ \{ j_t^{L\pm,A}(x), j_t^{L\pm,B}(y) \}_P = \varepsilon^{AB}_{\ C} j_t^{L\pm,C}(x) \delta(x - y) \]

\[ \{ j_t^{L\pm,A}(x), j_x^{L\pm,B}(y) \}_P = \varepsilon^{AB}_{\ C} j_x^{L\pm,C}(x) \delta(x - y) + (1 + C) \delta^{AB} \partial_x \delta(x - y) \]

\[ \{ j_x^{L\pm,A}(x), j_x^{L\pm,B}(y) \}_P = -C \varepsilon^{AB}_{\ C} j_t^{L\pm,C}(x) \delta(x - y) \]

The current algebra is deformed due to the improvement.

Is Yangian algebra still realized?

(non-trivial question)
A pair of $SU(2)_L$ Yangian algebras

\[
\{Q^{L,A}_{(0)}, Q^{L,B}_{(0)}\}_P = \varepsilon^{AB}_{\phantom{AB}C} Q^{L,C}_{(0)}
\]

\[
\{Q^{L,A}_{(0)}, Q^{L,B}_{(1)}\}_P = \varepsilon^{AB}_{\phantom{AB}C} Q^{L,C}_{(1)}
\]

\[
\{Q^{L,A}_{(1)}, Q^{L,B}_{(1)}\}_P = \varepsilon^{AB}_{\phantom{AB}C} \left[ Q^{L,C}_{(2)} + \frac{1}{12} Q^{L,C}_{(0)} Q^{L,D}_{(0)} Q^{L}_{(0)D} - C Q^{L,C}_{(0)} \right]
\]

Serre relations are also satisfied.

In summary,

Yangian algebra is realized even after the squashing.
Lax pairs and monodromy matrices

Two kinds of Lax pairs

\[ L_{\mu}^{L\pm}(\lambda_{L\pm}) \equiv \frac{1}{1 - \lambda_{L\pm}^2} \left[ j_{\mu}^{L\pm} - \epsilon_{\mu\nu} j^{\nu,L\pm} \right] \]

due to EOM and flatness condition

Monodromy matrices

\[ U^{L\pm}(\lambda_{L\pm}) = P \exp \left[ \int_{-\infty}^{\infty} dx \, L_{x}^{L\pm}(x; \lambda_{L\pm}) \right] \]

conserved

\[ \frac{d}{dt} U^{L\pm}(\lambda) = 0 \]

NOTE classical r-matrix is of rational type
The list of symmetries and integrable classes

<table>
<thead>
<tr>
<th>Global symm.</th>
<th>SU(2)$_L$</th>
<th>U(1)$_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hidden symm.</td>
<td>Yangian</td>
<td>quantum affine</td>
</tr>
<tr>
<td>Class (Lax pair)</td>
<td>rational</td>
<td>trigonometric</td>
</tr>
</tbody>
</table>

Squashed sigma models can be described as two different classes.

Hybrid integrable structure
3. Equivalence of two descriptions

[I. Kawaguchi and K.Y., PLB705 (2011) 251, 1107.3662]

Equivalence of monodromy matrices

\[
\begin{align*}
\tilde{g}_+^{-1} \cdot U^L_+ (\lambda_{L_+}) \cdot \tilde{g}_+ &= U^R (\lambda_R) \\
\tilde{g}_-^{-1} \cdot U^L_- (\lambda_{L_-}) \cdot \tilde{g}_- &= U^R (\lambda_R)
\end{align*}
\]

(Re \(z_R \geq 0\))

\(z_R = e^{-\lambda_R}\)

where \(\tilde{g}_\pm \equiv g_\infty \cdot e^{\pm i T^3 \lambda_R}\)

Rescaling of sl(2) generators

\[
\begin{align*}
T^\pm &\to e^{\pm \lambda_R} T^\pm &\text{for } U^L_+ (\lambda_{L_+}) \\
T^\pm &\to e^{\pm \lambda_R} T^\pm &\text{for } U^L_- (\lambda_{L_-})
\end{align*}
\]

(The derivation will be explained later)
The gauge equivalence of Lax pair

Start from a left Lax pair with + sign,

\[ L_{\pm}^{L+}(x; \lambda_{L+}) = \frac{1}{1 \pm \lambda_{L+}} g \left[ T^+ \left( 1 \mp i \sqrt{C} \right) J^- \pm T^- \left( 1 \pm i \sqrt{C} \right) J^+ + T^3(1 + C) J^3 \right] g^{-1} . \]

The gauge transformation is given by

\[
\left[ L_{\pm}^{L+}(x; \lambda_{L+}) \right]^g \equiv g^{-1} L_{\pm}^{L+}(x; \lambda_{L+}) g - g^{-1} \partial_{\pm} g \\
= -J_{\pm} + \frac{1}{1 \pm \lambda_{L+}} \left[ T^+ \left( 1 \mp i \sqrt{C} \right) J^- \pm T^- \left( 1 \pm i \sqrt{C} \right) J^+ + T^3(1 + C) J^3 \right] \\
= -\frac{\pm \lambda_{L+}}{1 \pm \lambda_{L+}} \left[ T^+ \left( 1 + \frac{i \sqrt{C}}{\lambda_{L+}} \right) J^- \pm T^- \left( 1 - \frac{i \sqrt{C}}{\lambda_{L+}} \right) J^+ + T^3 \left( 1 + \frac{C}{\lambda_{L+}} \right) J^3 \right] .
\]
By using the relation between the spectral parameters,

\[ \lambda_{L\pm} = \frac{\tanh \alpha}{\tanh \lambda_R}, \quad \text{(inverse relation)} \]

we obtain that

\[ \left[ L^L_\pm(x; \lambda_{L_+}) \right]^g = -\frac{\sinh \alpha}{\sinh(\alpha \pm \lambda_R)} \left[ T^+ e^{\lambda R} J^-_\pm + T^+ e^{-\lambda R} J^+_{\pm} + T^3 \frac{\cosh(\alpha \pm \lambda_R)}{\cosh \alpha} J^3_{\pm} \right]. \]

By rescaling the generators,

\[ T^\pm \rightarrow e^{\mp \lambda R} T^\pm \quad \text{for} \quad U^L_+(\lambda_{L_+}), \]

we can show that

\[ \left[ L^L_\pm(x; \lambda_{L_+}) \right]^g \simeq L^R_\pm(x; \lambda_R). \quad \text{(trigonometric Lax pair!)} \]

Thus we can show the equivalence at the monodromy matrix level,

\[ g^{-1}_\infty \cdot U^L_+(\lambda_{L_+}) \cdot g_\infty \simeq U^R(\lambda_R). \]
How to derive the spectral parameter relation

The relation is rewritten as

\[ z_R^2 = \frac{\lambda_{L \pm} - i\sqrt{C}}{\lambda_{L \pm} + i\sqrt{C}} \]  

(Möbius trans.)

The data of monodromy matrix expansions

<table>
<thead>
<tr>
<th>Charges \ Monodromies</th>
<th>( U^R(\lambda_R) )</th>
<th>( U^{L+}(\lambda_{L+}) )</th>
<th>( U^{L-}(\lambda_{L-}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q^{R,3}<em>{(0)} ), ( Q^{R,-}</em>{(1)} ), ( \tilde{Q}^{R,+}_{(1)} )</td>
<td>0</td>
<td>+i\sqrt{C}</td>
<td>+i\sqrt{C}</td>
</tr>
<tr>
<td>( Q^{R,3}<em>{(0)} ), ( Q^{R,+}</em>{(1)} ), ( \tilde{Q}^{R,-}_{(1)} )</td>
<td>\infty</td>
<td>-i\sqrt{C}</td>
<td>-i\sqrt{C}</td>
</tr>
<tr>
<td>( Q^{L,a}<em>{(0)} ), ( Q^{L,a}</em>{(1)} )</td>
<td>±1</td>
<td>\infty</td>
<td>\infty</td>
</tr>
<tr>
<td>local charges</td>
<td>±e^{\alpha}, ±e^{-\alpha}</td>
<td>±1</td>
<td>±1</td>
</tr>
</tbody>
</table>

Note: \( U^R \) is expanded in terms of \( z_R \)

- \( U^R(\lambda_R) \)
- \( U^{L\pm}(\lambda_{L\pm}) \)

might be a bit surprising,

1) \( SU(2)_L \) Yangian algebra can be reproduced from \( U^R(\lambda_R) \)
2) Quantum affine algebra can be reproduced from \( U^{L\pm}(\lambda_{L\pm}) \)

has been discussed already
What is the geometrical meaning of the relation?

The space of spectral parameter in the trigonometric description

There are four poles in the trigonometric Lax pair.

The position of poles depends on the value of $C$. 

a) For $C > 0$

b) For $-1 < C < 0$
According to the map \( z_R = e^{-\lambda_R} \)

C > 0

Riemann sphere with four punctures
The space of spectral parameter in the rational description

Two Riemann spheres with two punctures

The two Riemann spheres are joined on the cut.

This is regarded as a Riemann sphere with four punctures
The spaces of spectral parameters are identical

The imaginary axis on the $z_R$-plane corresponds to the cut on $\lambda_{L \pm}$-plane.

A trigonometric description = a pair of rational descriptions

$C \to 0$ limit: The cut shrinks to a point ($SU(2)_R$ Yangian point)

Inversely speaking, the breaking of $SU(2)_R$ opens up the cut.
4. Summary & Discussions
Summary

We have discussed the classical integrable structure of squashed sigma models.

- SU(2)_L Yangians & quantum affine algebra
- Two descriptions
  1) rational one
  2) trigonometric one
  → Hybrid classical integrable structure
- Equivalence of two descriptions at the monodromy matrix level

Note  The present argument is applicable to warped AdS_3 at classical level.

Application to AdS/condensed matter physics?  [D’Hoker-Kraus, 2009]
Application to warped AdS_3/dipole CFT_2?  [El-Showk-Guica, Song-Strominger, 2011]
Other directions

1) Schrödinger sigma models
   The same argument is applicable.  
   → q-deformed Poincare symmetry  
   affine extension?  
   [Kawaguchi and K. Y., 1109.0872] 
   Kawaguchi’s poster

2) Squashed Wess-Zumino-Novikov-Witten model
   The trigonometric description becomes rational
   at certain values of the coefficient of WZ term
   (in progress)
   c.f., [Fateev, NPB473 (1996) 509]

3) Other integrable deformations of $S^3$
   “trigonometric & trigonometric”  “rational & elliptic” etc.
   c.f., [Fateev, NPB473 (1996) 509]

4) Half line case
   twisted quantum affine, twisted Yangian?
   c.f., [MacKay-Short, hep-th/0104212]
   Regelskis’ poster
Thank you!
Backup
One may see a larger hidden structure by deforming integrable systems.

EX

XX model $\rightarrow$ XXZ model

Hamiltonian of XXZ model

$$H_{XXZ} = -J \sum_j \left[ S_{j+1}^x S_j^x + S_{j+1}^y S_j^y + \Delta S_{j+1}^z S_j^z \right]$$

$$\Delta = 1 : \text{XXX} \quad \text{SU(2) symmetry (isotropic)}$$

$$\Delta \neq 1 : \text{XXZ} \quad \text{U(1) symmetry (uniaxial)}$$

But the remaining U(1) is enhanced to $q$-deformed SU(2) $U_q(su(2))$

The $q$-deformed SU(2) is further enhanced to quantum affine algebra $U_q(\widehat{su(2)})$

Symmetry is enhanced by integrable deformation
The universality classes of classical integrable systems

1) Rational class - XXX model (No deformation parameter)
   Hidden symmetry: Yangian algebra

2) Trigonometric class - XXZ model (1 deformation parameters)
   Hidden symmetry: quantum affine algebra

3) Elliptic class - XYZ model (2 deformation parameters)
   Hidden symmetry: elliptic algebra

[Belavin-Drinfeld, 1982]
BIZZ construction

Assume that we have a flat conserved current \( j_\mu \)

Let’s introduce the covariant derivative:

\[
D_\mu = \partial_\mu - j_\mu
\]

satisfies:

\[
\partial_\mu D_\mu = D_\mu \partial_\mu \]

\[
\epsilon^{\mu\nu} D_\mu D_\nu = 0
\]

\[
\epsilon^{\mu\nu} (\partial_\mu j_\nu - j_\mu j_\nu) = 0
\]

With the covariant derivative, one can construct an infinite number of non-local charges recursively.

NOTE If there is a flat conserved current, then \( M \) is not needed to be symmetric.
Let’s take the Noether current as the 0th current:

\[ J_{(0)\mu} = j_\mu = D_\mu \chi(0) \quad \text{\rightarrow} \quad \partial^\mu J_{(0)\mu} = 0 \quad \text{Conserved by definition.} \]

\( (\chi(0) = -1) \)

\[ J_{(0)\mu} = \epsilon_{\mu\nu} \partial^\nu \chi(1) \]

\( \epsilon(x - y) \equiv \theta(x - y) - \theta(y - x) \)

\[ \chi(1)(x) = \frac{1}{2} \int dy \, \epsilon(x - y) J_{(0)t}(y) \]

Then the next current is defined as

\[ J_{(1)\mu} \equiv D_\mu \chi(1) \quad : \text{conserved} \]

(\therefore \quad \partial^\mu J_{(1)\mu} = \partial^\mu D_\mu \chi(1) = D_\mu \partial^\mu \chi(1) = \epsilon^{\mu\nu} D_\mu J_{(0)\nu} \]

\[ = \epsilon^{\mu\nu} D_\mu D_\nu \chi(0) = 0 \]

Repeat the same step \quad \text{\rightarrow} \quad \text{Infinite number of non-local charges}
$q$-deformed Poincare algebra

The Poisson brackets we obtained:

\[
\{ Q_{R,+}, Q_{R,-} \}_P = -Q_{R,2},
\]

\[
\{ Q_{R,+}, Q_{R,2} \}_P = -Q_{R,+} \cosh \left( \frac{\sqrt{C}}{2} Q_{R,-} \right),
\]

\[
\{ Q_{R,-}, Q_{R,2} \}_P = \frac{2}{\sqrt{C}} \sinh \left( \frac{\sqrt{C}}{2} Q_{R,-} \right).
\]

By rescaling the charge as $Q_{R,+} \rightarrow \frac{\sqrt{C}}{2} Q_{R,+}$

$q$-deformed Poincare algebra:

\[
\{ Q_{R,+}, Q_{R,-} \}_P = -\frac{\sqrt{C}}{2} Q_{R,2},
\]

\[
\{ Q_{R,+}, Q_{R,2} \}_P = -Q_{R,+} \cosh \left( \frac{\sqrt{C}}{2} Q_{R,-} \right),
\]

\[
\{ Q_{R,-}, Q_{R,2} \}_P = \frac{2}{\sqrt{C}} \sinh \left( \frac{\sqrt{C}}{2} Q_{R,-} \right).
\]
The classical action revisited

\[ S = \frac{1}{1 + C} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \, \eta^{\mu \nu} \, \text{Tr} (j^L_{\mu} + j^L_{\nu}) \]

The classical action can be expressed in terms of the improved currents.

Dipole-like form!
Warped $\text{AdS}_3$ $\equiv$ a double Wick rotation of squashed $S^3$

$$S^3 \rightarrow \text{AdS}_3 , \quad SU(2) \rightarrow SL(2,R)$$

1) space-like warped $\text{AdS}_3$ : $\theta \rightarrow i\sigma, \phi \rightarrow i\mu, \psi \rightarrow \tau$

$$ds^2 = \frac{L^2}{4} \left[- \cosh^2 \sigma d\tau^2 + d\sigma^2 + (1 + C)(d\mu + \sinh \sigma d\tau)^2\right]$$

2) time-like warped $\text{AdS}_3$ : $\theta \rightarrow i\sigma, \phi \rightarrow \tau, \psi \rightarrow i\mu$

$$ds^2 = \frac{L^2}{4} \left[-(1 + C)(d\tau - \sinh \sigma d\mu)^2\right] + d\sigma^2 + \cosh^2 \sigma d\mu^2$$

The difference between warped $\text{AdS}_3$ and squashed $S^3$ is just signature at least at classical level.
Equivalence of two descriptions

Two types of Lax pair coexist in squashed sigma models.

In order to understand the relation between them, it would be helpful to compare the present case with principal chiral models.

\[
\begin{array}{|c|c|}
\hline
\text{SU(2) principal chiral models} & \text{Sigma models on squashed } S^3 \\
\hline
\text{SU(2)}_L \text{ symmetry} & \text{SU(2)}_L \text{ symmetry} \\
\text{SU(2)}_R \text{ symmetry} & \text{q-deformed SU(2)}_R \text{ symmetry} \\
\text{rational} & \text{trigonometric} \\
\hline
\end{array}
\]

deformation (squashing)
Let us see the relation between the two descriptions.

SU(2) principal chiral models $(j^R_\mu = dg \cdot g^{-1}, \ j^L_\mu = g^{-1}dg)$

$C = 0$

Two rational descriptions are equivalent.

The relationship: $j^R_\mu = g^{-1}j^L_\mu g$ (left-right symm.)

deformation (squashing)

$C \neq 0$

A similar relation holds even after the squashing

$$j^{R,3}_\mu = -2\text{Tr} \left(T^3 g^{-1} j^L_\mu + g\right)$$

$$j^{R,\pm}_\mu = -2e^{\gamma \chi} \text{Tr} \left(T^{\pm} g^{-1} j^L_\mu + g\right)$$

Non-local map


NOTE $j^L_\mu$ corresponds to affine generators.